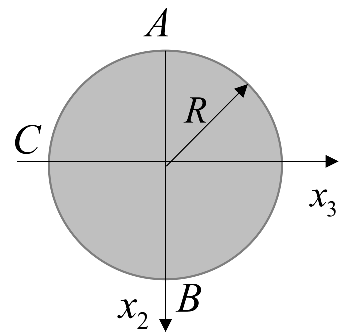
Exercise 1: A solid cylinder of radius R, is subjected to a bending moment M_f and a torque M_t (R=100mm, M_f=3000Nm; M_t=4000Nm).



Determine the components of the deviatoric stress tensor s_{ij} and the effective stress σ_e at points *A, B* and *C*. Comment on the results.

Note that the stresses due to bending and torque are given by $\sigma = M_f x_2 / I$; $\tau = M_t r / I_p$ with I, I_p indicating the moments of inertia around the bending axis and the polar moment of inertia, respectively. For a circular cross-section we have $I_p = 2I = \pi R^4 / 2$.

Solution

We consider here that bending is around the x_3 axis, the traction along the x_1 and the torsion around x_1. According to the beam theory, the stresses in the section are not uniform.

The maximum normal stress due to bending is at point A (positive) and B (negative). Along the x_3 axis the normal stress is zero everywhere. As for the maximum shear it is tangent all along the perimeter of the section.

At point *A* the stress are,

$$\sigma_{11} = \frac{M_f R}{\pi R^4 / 4} = \frac{4M_f}{\pi R^3} = 3.82 MPa$$

$$\sigma_{13} = \frac{M_t R}{\pi R^4 / 2} = \frac{2M_t}{\pi R^3} = 2.55 MPa, \quad \sigma_{22} = \sigma_{33} = \sigma_{12} = \sigma_{23} = 0 \quad \Rightarrow [\sigma] = \begin{pmatrix} 3.82 & 0 & 2.55 \\ 0 & 0 & 0 \\ 2.55 & 0 & 0 \end{pmatrix}$$

With $s_{ij} = \sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk} \Rightarrow [s] = \begin{pmatrix} 2\frac{3.82}{3} & 0 & 2.55 \\ 0 & -\frac{3.82}{3} & 0 \\ 2.55 & 0 & -\frac{3.82}{3} \end{pmatrix} = \begin{pmatrix} 2.55 & 0 & 2.55 \\ 0 & -1.27 & 0 \\ 2.55 & 0 & -1.27 \end{pmatrix}$

The equivalent stress is,

$$\sigma_e^2 = \frac{1}{2}\left[(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2 + 6\sigma_{12}^2 + 6\sigma_{23}^2 + 6\sigma_{31}^2\right]$$

$$= \frac{1}{2}\left[2\sigma_{11}^2 + 6\sigma_{31}^2\right] \Rightarrow \sigma_e = 5.84MPa$$

<u>At point *B* the stresses are:</u>

$$\sigma_{11} = -\frac{M_f R}{\pi R^4/4} = -\frac{4M_f}{\pi R^3} = -4\frac{3\cdot 10^3}{3.14\cdot 10^{-3}} = -3.82MPa$$

$$\sigma_{13} = -\frac{M_t R}{\pi R^4/2} = -\frac{2M_t}{\pi R^3} = -2.55MPa, \quad \sigma_{22} = \sigma_{33} = \sigma_{12} = \sigma_{23} = 0 \quad \Rightarrow [\sigma] = \begin{pmatrix} -3.82 & 0 & -2.55 \\ 0 & 0 & 0 \\ -2.55 & 0 & 0 \end{pmatrix}$$

The negative sign in σ_{31} is because the stress points towards the negative direction of x_3.

The negative sign in σ_{11} is because the stress points towards the negative direction of x_1.

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk} \Rightarrow [s] = \begin{pmatrix} -2.55 & 0 & -2.55 \\ 0 & 1.27 & 0 \\ -2.55 & 0 & 1.27 \end{pmatrix}$$

$$\sigma_e^2 = \frac{1}{2}\left[(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2 + 6\sigma_{12}^2 + 6\sigma_{23}^2 + 6\sigma_{31}^2\right]$$

$$= \frac{1}{2}\left[2(\sigma_{11})^2 + 6\sigma_{31}^2\right] \Rightarrow \sigma_e = 5.84MPa$$

At point *C* the stress are:

$$\sigma_{11} = 0$$

$$\sigma_{12} = -\frac{M_t R}{\pi R^4 / 2} = -\frac{2M_t}{\pi R^3} = -2.55 MPa, \quad \sigma_{22} = \sigma_{33} = \sigma_{13} = \sigma_{23} = 0 \quad \Rightarrow [\sigma] = \begin{pmatrix} 0 & -2.55 & 0 \\ -2.55 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk} \Rightarrow [s] = \begin{pmatrix} 0 & -2.55 & 0 \\ -2.55 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\sigma_e^2 = \frac{1}{2}\left[\left(\sigma_{11} - \sigma_{22}\right)^2 + \left(\sigma_{22} - \sigma_{33}\right)^2 + \left(\sigma_{33} - \sigma_{11}\right)^2 + 6\sigma_{12}^2 + 6\sigma_{23}^2 + 6\sigma_{31}^2 \right]$$

$$= \frac{1}{2}\left[6\sigma_{31}^2 \right] \Rightarrow \sigma_e = 4.42 MPa$$

Comments:

The negative sign of the normal stress at point *B* does not influence the equivalent stress. The same is true for the shear stress because their values are squared.

We also see the important role of the shear stress in the overall value on equivalent stress.

Exercise 2: Show that for the Prandtl – Reuss plastic strain increment relations,

$$d\varepsilon_{ij}^{p} = \frac{3}{2}\frac{d\varepsilon_p}{\sigma_e} s_{ij}$$

the plastic work increment can be expressed as

$$dW^p = \sigma_e d\varepsilon_p$$

Where σ_e is the equivalent stress and $d\varepsilon_p$ the equivalent strain increment.

Solution

By definition the plastic work is,

$$dW^p = \sigma_{ij} d\varepsilon_{ij}^{p} \qquad \text{(C.35c)}$$

In this expressions, we replace the stresses by the deviatoric and volumetric components,

$$\sigma_{ij} = s_{ij} + \frac{1}{3}\delta_{ij}\sigma_{kk}$$

and take into account that $d\varepsilon_{ii}^{p} = 0$,

$$dW^p = \left(s_{ij} + \frac{1}{3}\delta_{ij}\sigma_{kk}\right) d\varepsilon_{ij}^{p} = s_{ij} d\varepsilon_{ij}^{p} + \frac{1}{3}\delta_{ij} d\varepsilon_{ij}^{p}\sigma_{kk}$$

$$= s_{ij} d\varepsilon_{ij}^{p} + \frac{1}{3} d\varepsilon_{ii}^{p}\sigma_{kk} = s_{ij} d\varepsilon_{ij}^{p}$$

The Prandtl – Reuss are

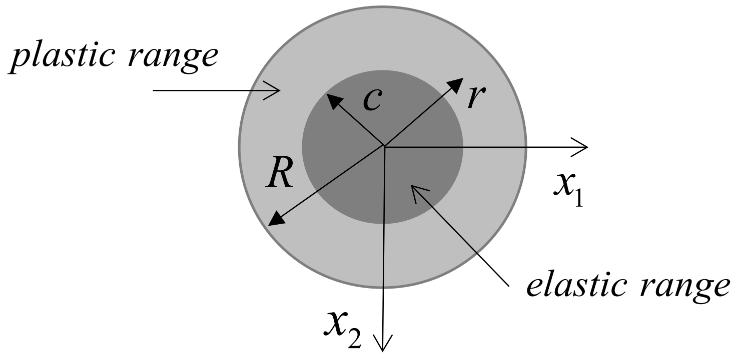
$$d\varepsilon_{ij}^{p} = \frac{3}{2}\frac{d\varepsilon_p}{\sigma_e} s_{ij} \qquad \text{(C.33b)}$$

We replace the strains from (C.33b) in the expression for the work,

$$dW^p = \frac{3}{2}\frac{d\varepsilon_p}{\sigma_e} s_{ij}s_{ij} = \frac{3}{2}\frac{d\varepsilon_p\sigma_e}{\sigma_e^2} s_{ij}s_{ij} = \sigma_e d\varepsilon_p \frac{3}{2}\frac{s_{ij}s_{ij}}{\sigma_e} = \sigma_e d\varepsilon_p$$

We have seen in Example 3 of the Appendix C that, $\sigma_e^2 = \frac{3}{2} s_{ij}s_{ij}$

Exercise 3: An elastic perfectly plastic circular shaft of radius R is subjected to a torsional moment M_t at its ends. Determine the M_t at first yield and M_t for which there is an inner elastic core of radius *c*. Give a schematic of the stress distribution in both cases and comment on the results. The yield stress of the material in shear is *k*.



Solution

The maximum stress in the shaft, according to the linear analysis, is at $r = R$ it is zero at the center and varies linearly from the center. Thus, at a distance *c* the shear stress distribution is given by,

$$\sigma_{12} = kr/c \quad for \quad 0 \le r \le c$$

In the plastic range we have for perfect plasticity,

$$\sigma_{12} = k \quad for \quad c \le r \le R$$

Using these expressions, the torque applied on the section with an elastic core is,

$$M_t = \int_0^c \left(\frac{kr}{c}\right)(2\pi r dr) r + \int_c^R (k)(2\pi r dr) r = 2\pi \int_0^c \left(\frac{kr^3}{c}\right) dr + 2\pi \int_c^R \left(kr^2\right) dr = 2\pi \frac{kc^4}{4c} + \frac{2\pi k}{3}\left(R^3 - c^3\right)$$

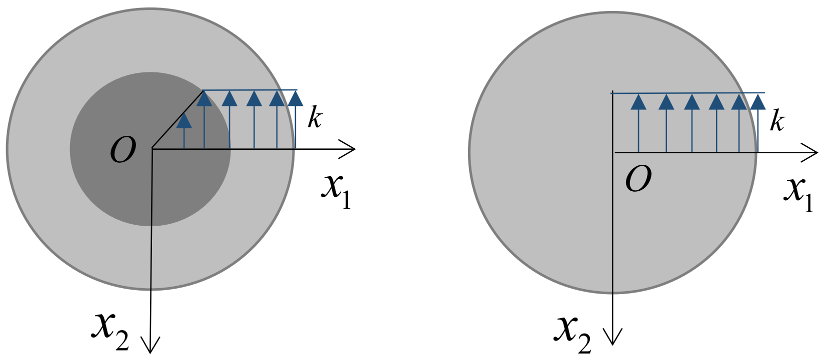
$$= \frac{\pi k c^3}{2} + \frac{2\pi k}{3}\left(R^3 - c^3\right) = \frac{2\pi k}{3}\left(R^3 - c^3/4\right)$$

At first yield we have $c = R \;\Rightarrow\; M_t = \dfrac{\pi k R^3}{2}$.

At full yield we have $c = 0 \;\Rightarrow\; M_t = \dfrac{2\pi k R^3}{3} = \dfrac{4}{3}\left(\dfrac{\pi k R^3}{2}\right)$.

<u>Schematics of the stress distribution</u>

Partial yield on the left and full yield on right



<u>Comment:</u>

The full yield is not realistic at the center of the shaft since the fiber though the center is not stressed. Namely, it should be zero at the center.